High Order Modes in a Spherical Fabry–Perot Resonator

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Abstract—**The accuracy of the approximate solution to the wave equation in the "large aperture" case was investigated. The measured distribution of energy in the various transverse modes corresponded to the Laguerre–Gaussian solutions; resonant frequencies, however, deviated from those predicted by the approximate theory by as much as 2 percent for high radial mode numbers. Two first order perturbation calculations, including a neglected term in the wave equation and the nonsphericity of constant phase surfaces, yielded resonant frequencies in agreement with experiment.**

I. INTRODUCTION

THE USE of a Fabry–Perot structure as a laser resonator [1], along with its obvious relation to the beam waveguide for transmission of very short wavelength microwave power [2], has provided the impetus for developing an electromagnetic theory of its operation [3]–[5]. Measurements at microwave frequencies have often been used to verify the theory and analyze the effect of different parameters [6].

Generally, closed form solutions to the Fabry–Perot problem are the result of an approximation. In systems with "large aperture," i.e., where the radial extent of the mirrors is large enough to reflect all but a negligible portion of beam energy, diffraction is neglected and a wave analysis of the resonator is carried out as follows: a solution of the form $u = \exp(-jkz)\psi(r,\phi,z)$ is substituted into the wave equation $\nabla^2 u + k^2 u = 0$, and the term $\partial^2\psi/\partial z^2$ is neglected because ψ is assumed to be a slowly varying function of z. With this approximation the solutions for the fundamental and higher order modes are the familiar Laguerre–Gaussian functions, which satisfy orthogonality relations like modes in a normal resonator.

We have investigated a "large aperture" spherical mirror Fabry–Perot resonator at X-band frequencies. In this paper we report the frequency-shifting effects of including $\partial^2\psi/\partial z^2$ and the exact shape of constant phase surfaces in the wave equation as first order perturbations, and compare the predicted resonant frequencies with experiment.

II. APPROXIMATE SOLUTION TO THE WAVE EQUATION

This treatment follows that of Kogelnik and Li [7]. A component of electric field u satisfies the scalar wave equation

$$\nabla^2 u + k^2 u = 0 \tag{1}$$

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where $k = 2\pi/\lambda$ is the propagation constant. Substitution into (1) of the trial function $u = \exp(-jkz)\psi(r,\phi,z)$ yields the equation

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} = 2jk\frac{\partial\psi}{\partial z}. \tag{2}$$

At this point the assumption is made that the function ψ varies so slowly with z that the second derivative with respect to z can be neglected. With this assumption the solution is

$$\psi_p{}^l = \left(\sqrt{2}\frac{r}{w}\right)^l L_p{}^l\left(\frac{2r^2}{w^2}\right)\frac{w_0}{w}\exp(-r^2/w^2) \cdot \exp\langle j[(2p+l+1)\tan^{-1}az - (r^2/w^2)az + l\phi]\rangle \tag{3}$$

where $L_p{}^l$ is the generalized Laguerre polynomial, p and l are the radial and azimuthal mode numbers, respectively, and w_0 is the beam waist defined by the equation

$$w_0{}^2 = \frac{\lambda}{2\pi}[d(2R-d)]^{1/2}. \tag{4}$$

Summarizing the remaining terms, we have the following:

- d spacing between the mirrors (mirrors are at $z = \pm d/2$);
- R radius of curvature of the mirrors;
- a $\lambda/\pi w_0{}^2$;
- w $w_0[1 + a^2z^2]^{1/2}$;
- ϕ azimuthal coordinate in the cylindrical coordinate system.

The total wave function $u_p{}^l$ is just $\exp(-jkz)$ times the expression (3). The condition of resonance is that the phase shift from one mirror to the other be an integral multiple of π; i.e.,

$$kd - 2(2p+l+1)\tan^{-1}\left(\frac{ad}{2}\right) = q\pi \tag{5}$$

where q is the axial mode number. Since the frequency rather than wavenumber is observed experimentally, we rewrite formula (5) in terms of f:

$$f = \frac{c}{2d}\left[q + \frac{2}{\pi}(2p+l+1)\tan^{-1}\left(\frac{ad}{2}\right)\right]. \tag{6}$$

Note that the phase shift has been evaluated on the axis ($r = 0$). Since the phase fronts are assumed spherical, as are the mirror surfaces, the phase remains constant over the entire mirror. Actually, like the neglect of $\partial^2\psi/\partial z^2$,

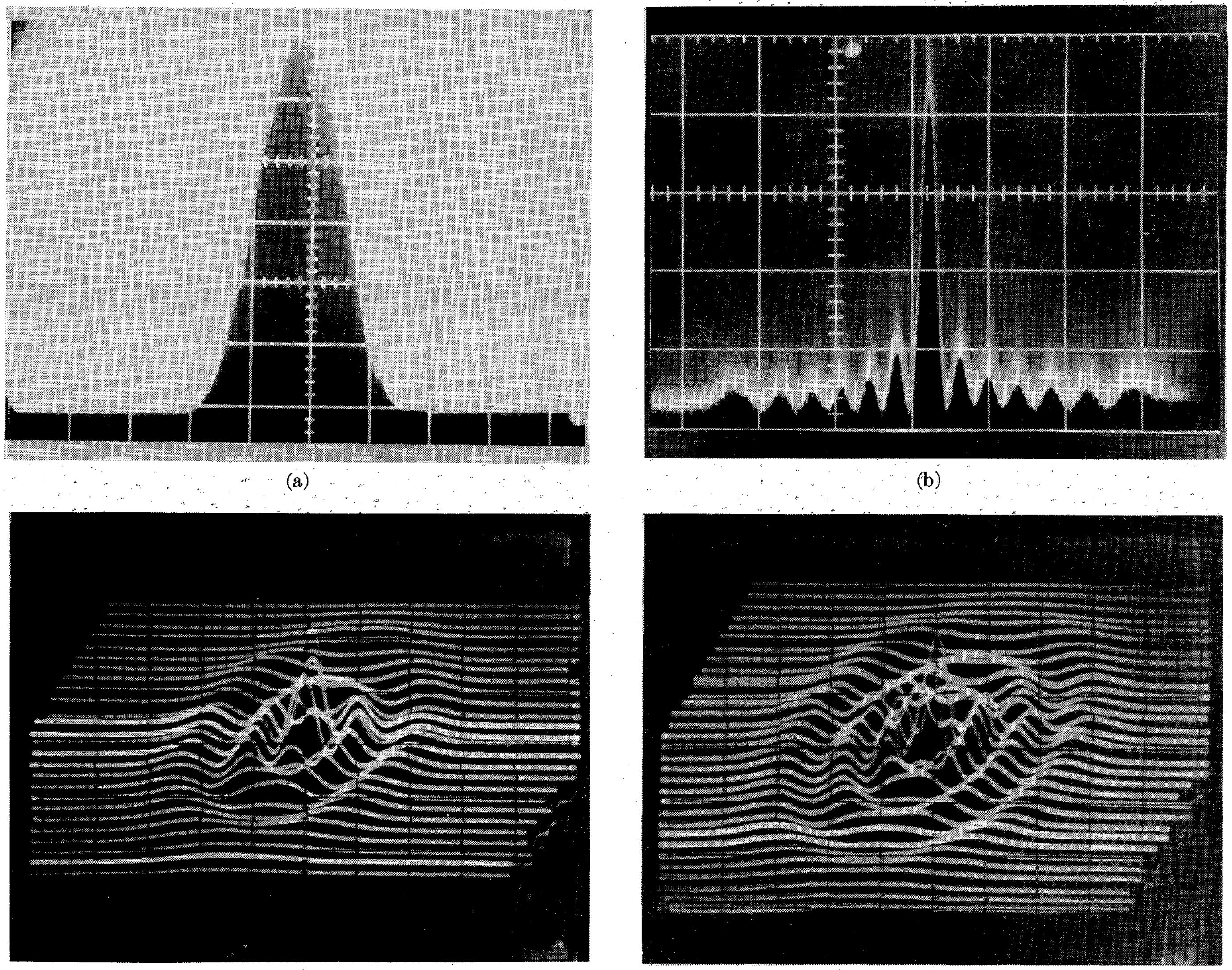

Fig. 1. (a) and (b) Radial scans in the midplane of the energy distribution of the fundamental (TEM_{00}) and the TEM_{06} mode, respectively. (b) and (c) Three dimensional oscilloscope representations of the energy distribution in the midplane of the TEM_{02} and TEM_{03} modes, respectively.

this is an approximation which we shall treat as a perturbation on the approximate solution.

III. EXPERIMENTAL APPARATUS

The mirrors were made of aluminum with a 45-cm diameter and 61-cm radius of curvature. Signals were observed in transmission, with energy coupled in and out of the cavity through identical apertures (0.25-in diameter) in a coupling diaphragm on the axis. In this way modes with zero energy on the axis ($l \neq 0$) were not excited, so that frequency degeneracies for identical combinations of $(2p + l + 1)$ were avoided. Resonant frequencies were measured with a direct reading frequency meter and the energy distribution of the various modes was measured by the absorptive probe technique described in [8]. The probe was a 3-mm-diameter sphere of Eccosorb® which gave excellent spatial resolution.

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Transverse modes of the type TEM_{0p} were observed for p as high as 12 at very close mirror spacing. Measurements were always made in the midplane of the device to insure that the straight line sweep of the probe remained on a surface of constant phase. Cross-sectional scans and three-dimensional oscilloscope simulations of the energy distribution of several modes in the midplane are shown in Fig. 1.

The beam waists w_0 calculated from the radial positions of the zeros of the measured distribution function agreed with the expression in (4) to within 5–10 percent. The accuracy of peak height measurements is estimated to be 10–20 percent. Resonant frequencies, however, can easily be measured to one part in 10^4 (1 MHz in 10 GHz) so that small discrepancies between results of theory and experiment would be most readily observable in the frequency measurement. Discrepancies did exist; the measured frequency exceeded the theoretical frequency in all modes by an amount depending on the radial mode number

p. Since the maximum observed frequency discrepancy for any mode was less than 2 percent, we applied a first order perturbation correction to the theoretical solution.

IV. PERTURBATION ANALYSES

Two assumptions were made in obtaining the Laguerre–Gaussian solutions to the wave equation in cylindrical coordinates: 1) the term $\partial^2\psi/\partial z^2$ was assumed negligible, and 2) the surfaces of constant phase were assumed spherical, even at large distances from the axis. Using the Laguerre–Gaussian solution we calculated the effects on the resonant frequency of: 1) a perturbation of the boundary from the constant phase surface of the theoretical solution to that of the actual mirror, and 2) a perturbation of the differential operator, namely, the neglected term $\partial^2\psi/\partial z^2$.

Since these perturbations are to be calculated only to first order they will be treated separately and their effects added to obtain the total frequency shift.

A. Perturbation of Boundary Surface

Fig. 2 shows that the constant phase surfaces of the theoretical problem are mode dependent and that at larger distances from the axis they are not spherical. Intuitively, one would expect an experimental frequency lower than the theoretical, since the actual mirror surfaces are farther apart than the theoretical ones. (At very large radii the theoretical surfaces are farther apart, but there is negligible energy at such large radii.)

Calling the theoretical constant phase surface S^1 we have a wave function u which obeys the equation

$$\nabla^2 u + k_u^2 u = 0 \tag{7}$$

and is 0 on S^1. Let the actual mirror surface be S and assume a new wave function v, which is 0 on S and obeys the equation

$$\nabla^2 v + k_v^2 v = 0. \tag{8}$$

Green's theorem states that for any two functions

$$\int_V (u\nabla^2 v - v\nabla^2 u)\, d^3X = \int_S \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dA \tag{9}$$

where V is the volume enclosed by the surface S and $\partial/\partial n$ is the normal derivative.[1] Substituting from the wave equation and making the simplest possible approximation, namely, $v = u$, we have

$$k_u^2 - k_v^2 \simeq \frac{\int_S [u(\partial u/\partial n)]\, dA}{\int_V u^2\, d^3X}. \tag{10}$$

[1] The total surface S consists of the mirrors plus the cylindrical surface $r = R_{\text{mirror}}$, but for "large apertures" all functions and derivatives are zero for $r \geq R_{\text{mirror}}$, so only the mirror surface contributes to the integral.

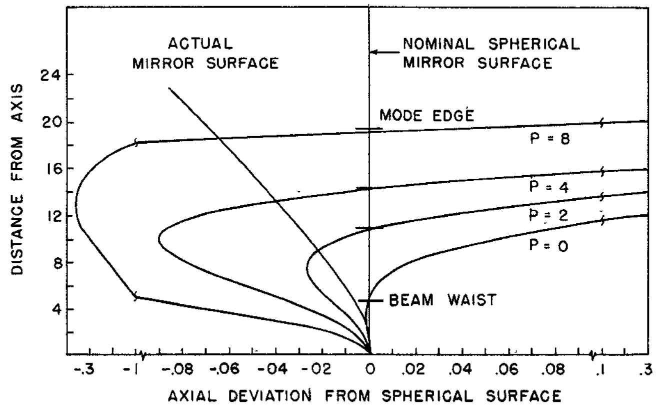

Fig. 2. Axial deviation from the designed spherical surface for constant phase surfaces of various theoretical modes, as well as for the mirror itself. The hash marks indicating the mode edges are the radial positions beyond which the function decreases monotonically from 10^{-6} of its magnitude on the axis.

Since S and S^1 are displaced only slightly from each other we expand u and $\partial u/\partial n$ in a Taylor's series about S^1, keeping only the highest order term and neglecting the z dependence of ψ, to get

$$k_u^2 - k_v^2 = \frac{k_u^2 \int_{S^1} \Delta n \psi^2\, dA}{\frac{1}{2}d \int_{S^1} \psi^2\, dA}. \tag{11}$$

Since the ψ's may be normalized the integral in the denominator may be taken to be unity, and for small differences between k_u and k_v we have

$$\frac{\Delta f}{f} \simeq \frac{\int_{S^1} \Delta n \psi^2\, dA}{d} = \frac{\int_{S^1} \Delta z \cos\theta \psi^2\, dA}{d} \tag{12}$$

where the integral is taken over both mirrors, and θ is the angle between the axial direction and the normal to the mirror surface.

An independent check of this approximate theory was made by axial displacement of the coupling diaphragm. Here the formula for frequency shift becomes

$$\frac{\Delta f}{f} = \frac{\Delta z}{d} \int_{0\,(\text{one mirror})}^{R_d} \cos\theta \psi^2 2r\, dr \tag{13}$$

where Δz and R_d are the axial displacement and radius of the diaphragm, respectively. Note that in this case the higher order modes have a larger portion of their energy on the unperturbed surface, implying a smaller frequency shift, whereas in the present problem the displacement between actual and theoretical mirror surfaces is larger for higher modes, implying a larger frequency shift. Fig. 3 gives the results of this check.

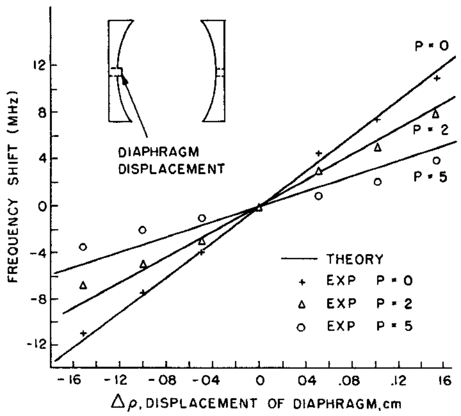

Fig. 3. Frequency shifts for the TEM_{00}, TEM_{02}, and TEM_{05} modes caused by axial displacement of the coupling diaphragm.

B. Perturbation of the Differential Operator

The equation for which an exact solution exists is

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} = 2jk \frac{\partial \psi}{\partial z}. \tag{14}$$

The solutions for $l = 0$ are

$$\psi_p{}^0 = L_p\left(\frac{2r^2}{w^2}\right)\frac{w_0}{w} \exp\,(-r^2/w^2)$$

$$\cdot \exp\left[j\left((2p+1)\tan^{-1}(az) - \frac{r^2}{w^2}az\right)\right]. \tag{15}$$

These solutions have two interesting properties which make them desirable to use in calculating the true function: 1) they are orthogonal, and 2) they are complete.

Making use of the latter we express the desired function ψ_p as a linear combination of the $\psi_p{}^0$ as follows:

$$\psi_p = \psi_p{}^0 + \sum_{n=0}^{\infty} C_{pn}(z)\psi_n{}^0(r,z). \tag{16}$$

Now we abbreviate the operator on the left-hand side of (14) by H^0, and let $\partial^2/\partial z^2 = H^1$. The problem now is to find the solutions ψ_p which satisfy

$$(H^0 + H^1)\psi_p = 2jk\frac{\partial \psi_p}{\partial z} \tag{17}$$

where

$$H_0\psi_p{}^0 = 2jk\frac{\partial \psi_p{}^0}{\partial z}. \tag{18}$$

Expanding ψ_p in terms of the $\psi_p{}^0$, substituting into (17), and neglecting second order terms gives

$$H^1\psi_p{}^0 = 2jk\sum_{n=0}^{\infty}\psi_n{}^0\frac{dC_{pn}(z)}{dz}. \tag{19}$$

At this point we make use of the orthogonality property of the $\psi_p{}^0$. Multiplying both sides of (19) by $\psi_m{}^{0*}r\,dr$ and integrating from 0 to ∞ gives

$$\int_0^\infty \psi_m{}^0 H^1\psi_p{}^0 r\,dr = 2jk\sum_{n=0}^{\infty}\frac{dC_{pn}(z)}{dz}\int_0^\infty \psi_m{}^{0*}\psi_n{}^0 r\,dr \tag{20}$$

where the integral

$$\int_0^\infty \psi_m{}^{0*}\psi_n{}^0 r\,dr = \begin{cases} 0, & m \neq n \\ \frac{1}{4}w_0{}^2, & m = n. \end{cases}$$

Thus we are left with differential equations for the $C_{pm}(z)$

$$\frac{w_0{}^2}{4}\frac{dC_{pm}(z)}{dz} = \frac{1}{2jk}\int_0^\infty \psi_m{}^{0*}H^1\psi_p{}^0 r\,dr. \tag{21}$$

The $\psi_p{}^0$ are known, and H^1 is just $\partial^2/\partial z^2$ so the equations to be solved are

$$\frac{w_0{}^2}{4}\frac{dC_{pm}(z)}{dz} = \frac{1}{2jk}\int_0^\infty \psi_m{}^{0*}\frac{\partial^2\psi_p{}^0}{\partial z^2} r\,dr. \tag{22}$$

For the reader's convenience an explicit expression for $\partial^2\psi_p{}^0/\partial z^2$, along with integral formulas of pertinent Laguerre polynomials, are given in the Appendix. With these, the integral in (22) can be evaluated and the differential equations for the C_{pm} integrated to yield

$$C_{pp-2} = \frac{a}{2k}\frac{p(p-1)}{4}(b_{pp-2} + jaz)$$

$$C_{pp-1} = \frac{a}{2k}p^2(b_{pp-1} + jaz)$$

$$C_{pp} = \frac{a}{2k}\frac{3p^2+3p+1}{2}(b_{pp} + jaz)$$

$$C_{pp+1} = \frac{a}{2k}(p+1)^2(b_{pp+1} + jaz)$$

$$C_{pp+2} = \frac{a}{2k}\frac{(p+1)(p+2)}{4}(b_{pp+2} + jaz) \tag{23}$$

where the b_{pm} are constants of integration. Note that only terms between $p-2$ and $p+2$ contribute to the new eigenfunction ψ_p. The requirement that the new ψ_p be orthogonal to first order makes $b_{ij} = -b_{ji}$ and hence all diagonal $b_{ii} = 0$. The other constants can be evaluated by the requirement that each term in the new wave function satisfy the resonance condition independently.

$$u_p = \exp\,(-jkz)\psi_p = \exp\,(-jkz)$$

$$\cdot\left[\psi_p{}^0(1 + C_{pp}) + \sum_{n=p-2;\,n\neq p}^{p+2} C_{pn}\psi_n{}^0\right]. \tag{24}$$

Thus for the diagonal term the phase is

$$kz - (2p+1)\tan^{-1}(az) - \tan^{-1}\left(\frac{a}{2k}\frac{3p^2+3p+1}{2}az\right) \tag{25}$$

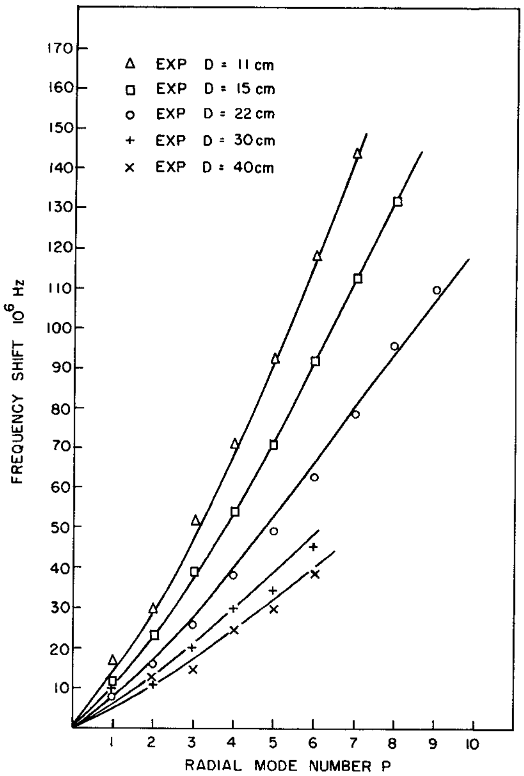

Fig. 4. Theoretical and experimental frequency shifts as a function of radial mode number for various mirror spacings. The shifts are given relative to that of the fundamental mode which was less than 3 MHz for all spacings.

and the resonance condition is

$$kd - 2(2p+1)\tan^{-1}\left(\frac{ad}{2}\right) - 2\tan^{-1} \cdot \left(\frac{a}{2k}\,\frac{3p^2+3p+1}{2}\,\frac{ad}{2}\right) = q\pi. \quad (26)$$

If we write $k = k_0 + \Delta k$ in the first term, the equation reduces to

$$\Delta k \simeq \frac{2}{d}\tan^{-1}\left(\frac{a}{2k_0}\,\frac{dp^2+3p+1}{2}\,\frac{ad}{2}\right) \quad (27)$$

where k_0 is the unperturbed eigenvalue.

Since $\Delta k = 2\pi\Delta f/c$ the frequency shift is

$$\Delta f = \frac{c}{\pi d}\tan^{-1}\left(\frac{a}{2k_0}\,\frac{3p^2+3p+1}{2}\,\frac{ad}{2}\right). \quad (28)$$

Note that the argument of the arc tangent is inherently positive, so that the frequency shift due to this perturbation is positive. In Fig. 4 the frequency shifts predicted by perturbation theory are compared with the measured frequency differences between simple theory and experiment, and we see that the two effects combine to give a good agreement.

V. CONCLUSIONS

A theoretical calculation was carried out and an experiment performed to examine the accuracy of the approximate solution to the wave equation in the "large aperture" case. The measured distribution of energy in the various transverse modes corresponded to the familiar Laguerre–Gaussian solutions within the experimental accuracy; however, the resonant frequency, which could be measured much more accurately, deviated from that predicted by the approximate theory by as much as 2 percent for high radial mode numbers. This discrepancy was traced to inadequacies in two additional assumptions made in solving the wave equation: 1) that the $\partial^2\psi/\partial z^2$ was negligible, and 2) that the surfaces of constant phase of the Laguerre–Gaussian solutions are spherical at large distances from the axis. Inclusion of the effects of two independent first order perturbation calculations yielded resonant frequencies in agreement with experiment. Although the experiment was done at microwave frequencies, there is nothing in the theory which precludes its use at other wavelengths.

APPENDIX

Direct differentiation of $\psi_p{}^0$ with respect to z, including the z dependence in w, yields after some algebraic manipulation

$$\frac{1}{2jk}\,\frac{\partial^2\psi_p{}^0}{\partial z^2}$$

$$= \frac{a^2}{2k}\Bigg\{-4jp(p-1)\,\frac{x^2}{X^2}\,\psi_{p-2}{}^0 \exp\,(4j\tan^{-1}x)$$

$$+ 2p\left[2(2p+1)\,\frac{x}{X^2} + j\left((4p+1)\,\frac{x^2}{X^2} - \frac{1}{X^2}\right)\right]\psi_{p-1}{}^0$$

$$\cdot \exp\,(2j\tan^{-1}x)$$

$$+ 2p\left[\frac{x^3}{X^2} - \frac{x}{X^2} - j\left(\frac{2x^2}{X^2}\right)\right]\frac{2r^2}{w^2}\,\psi_{p-1}{}^0 \exp\,(2j\tan^{-1}x)$$

$$+ 2(p+1)(2p+1)\left[\frac{-2x}{X^2} + j\left(\frac{1}{X^2} - \frac{x^2}{X^2}\right)\right]\psi_p{}^0$$

$$+ 2(p+1)\left[\frac{3x}{X^2} - \frac{x^3}{X^2} + j\left(\frac{3x^2}{X^2} - \frac{1}{X^2}\right)\right]\frac{2r^2}{w^2}\,\psi_p{}^0$$

$$+ \left[\frac{x^3}{X^2} - \frac{x}{X^2} + j\left(\frac{1}{4} - \frac{2x^2}{X^2}\right)\right]\left[\left(\frac{2r^2}{w^2}\right)^2\psi_p{}^0\right]\Bigg\}$$

where $x = az$ and $X = 1 + a^2z^2$. Repeated use has been made of the formula $x(d/dx)L_n(x) = nL_n(x) - nL_{n-1}(x)$ [9].

The following integrals of pertinent Laguerre polynomials are obtained from the formulas in [9], $L_n{}^\alpha(x) = L_n{}^{\alpha+1}(x) - L_{n-1}{}^{\alpha+1}(x)$, and

$$\int_0^\infty e^{-x}x^\alpha L_n{}^\alpha(x)L_m{}^\alpha(x)\,dx = \begin{Bmatrix} 0, & m \neq n \\ \Gamma(n+\alpha+1)/n!, & m = n \end{Bmatrix};$$

$$\int_0^\infty xe^{-x}L_pL_p\,dx = 2p+1$$

$$\int_0^\infty xe^{-x}L_{p-1}L_p\,dx = -p$$

$$\int_0^\infty x^2e^{-x}L_pL_p\,dx = 6p^2 + 6p + 2$$

$$\int_0^\infty x^2e^{-x}L_{p-1}L_p\,dx = -4p^2$$

$$\int_0^\infty x^2e^{-x}L_{p-2}L_p\,dx = p(p-1).$$

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A Unified Variational Solution to Microstrip Array Problems

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***Abstract*—A very general variational procedure is used to compute single or coupled microstrips under the quasi-TEM approximation. The capacitance model is found by means of a unique fundamental cell. The method is essentially an extension of Smith's [1], but may be used to study a wider variety of problems, such as nonuniform strip arrays, coplanar striplines, and broad-side coupled strips. Moreover, it is also possible to compute the coupling capacitance between nonadjacent strips.**

I. INTRODUCTION

It has been shown in [1], [2], that the capacitance model of single or coupled microstrip lines can be computed when capacitances of suitable "fundamental cells" are known. A fundamental cell is a single-strip rectangular region bounded by electric and/or magnetic walls. In [1] uniform strip arrays are computed by means of two fundamental cells, while in [2] the procedure is extended to nonuniform arrays by use of three fundamental cells. In the present paper a unique fundamental cell is used, including all the different types as special cases. In this way the computation procedure is unified and becomes particularly suitable for programming on a digital computer. Coplanar striplines as well as broad-side coupled strips can be calculated by this method; the saving in computer time is high (up to 70 percent) with respect to techniques based on optimization of the charge distribution, such as [7], and even higher with respect to relaxation techniques, such as [6]. This is due to the use of a variational method in the computations. The method is also suitable for computing the coupling capacitance between nonadjacent strips of a coplanar array.

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